

# Descent Functions and Random Young Tableaux

Ron M. Adin<sup>\*‡</sup>

Yuval Roichman<sup>†‡</sup>

submitted October 5, 1999 ; revised October 12, 2000

## Abstract

The expectation of the descent number of a random Young tableau of a fixed shape is given, and concentration around the mean is shown. This result is generalized to the major index and to other descent functions. The proof combines probabilistic arguments together with combinatorial character theory. Connections with Hecke algebras are mentioned.

## 1 Introduction

### 1.1 Background

In the late sixties Erdős and Turán have published a classical series of papers on random permutations. Since then there has been a resurgence of interest in probabilistic aspects of combinatorial parameters of permutations and related objects.

This paper deals principally with two classical combinatorial parameters: descent number and major index. These parameters were originally studied in the context of permutations. The study of the descent number of a permutation started with Euler; the major index has been introduced by MacMahon [M]. Foata-Schützenberger [F, FS], Garsia-Gessel [GG] and others carried out an extensive research of these parameters. The definitions of descent number and major index for permutations lead to definitions of

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<sup>\*</sup>Department of Mathematics and Computer Science, Bar-Ilan University, Ramat-Gan 52900, Israel. Email: [radin@math.biu.ac.il](mailto:radin@math.biu.ac.il)

<sup>†</sup>Department of Mathematics and Computer Science, Bar-Ilan University, Ramat-Gan 52900, Israel. Email: [yuvalr@math.biu.ac.il](mailto:yuvalr@math.biu.ac.il)

<sup>‡</sup>Both authors supported in part by the Israel Science Foundation and by internal research grants from Bar-Ilan University.

the same concepts for Young tableaux. These parameters on permutations – as well as on tableaux – play significant roles in algebraic combinatorics : the Solomon descent algebra [Re, Ch. 9], Schur functions [St, Ch. 7], and combinatorial character formulas [Ro2]. These concepts were also applied to sorting [K, Section 5.1] and card shuffling [DMP].

In this paper we study the distribution of these parameters for random Young tableaux of a given shape. Proofs of the main results are obtained by a combination of probabilistic arguments and combinatorial character theory.

## 1.2 Main Results

Let  $\lambda$  be a partition of  $n$  (For definitions of basic concepts see Section 2). We shall be concerned with *random* (standard Young) tableaux, assumed to be chosen uniformly with prescribed shape  $\lambda$ . A *descent* in a standard Young tableau  $T$  is an entry  $i$  such that  $i + 1$  is strictly south (and weakly west) of  $i$ . Denote the set of all descents in  $T$  by  $D(T)$ .

For any function  $f : \mathbf{N} \rightarrow \mathbf{R}$  we introduce the corresponding *descent function*  $d_f$  on standard Young tableaux :

$$d_f(T) := \sum_{i \in D(T)} f(i).$$

This concept generalizes two classical combinatorial parameters, the *descent number* and the *major index* (for tableaux) :

$$des(T) := \sum_{i \in D(T)} 1 ; \quad maj(T) := \sum_{i \in D(T)} i.$$

In this paper we prove

**Theorem 1.** *Let  $\lambda$  be a fixed partition of  $n$ , and let  $E_\lambda[d_f]$  be the expected value of a descent function  $d_f$  on random standard Young tableaux of shape  $\lambda$ . Then*

$$E_\lambda[d_f] = c(\lambda) \cdot \sum_{i=1}^{n-1} f(i).$$

where  $c(\lambda) := [(\binom{n}{2} - \sum_i \binom{\lambda_i}{2} + \sum_j \binom{\lambda'_j}{2})] / n(n-1)$ . (Here  $\lambda_i$  is the length of the  $i$ -th row in the Young diagram of shape  $\lambda$ , and  $\lambda'_i$  is the length of the  $i$ -th column.)

See Theorem 4.1 below.

Under mild conditions the descent function is concentrated around its mean. A function  $f : \mathbf{N} \rightarrow \mathbf{R}$  has *strictly polynomial growth* if there exist constants  $0 < c_1 < c_2$  and  $\alpha > 0$ , such that  $c_1 \leq \frac{f(n)}{n^\alpha} \leq c_2$  for  $n$  large enough.

**Theorem 2.** *Let  $0 < \delta < 1$  and  $\varepsilon > 0$  be fixed constants, and let  $\lambda$  be a partition of  $n$  with  $\lambda_1 \leq \delta n$ . Then for any function  $f$  with strictly polynomial growth, and for a random standard Young tableau  $T$  of shape  $\lambda$ ,*

$$d_f(T) = (1 + O(n^{-\frac{1}{2}+\varepsilon}))E_\lambda[d_f]$$

*almost surely (i.e., with probability tending to 1 as  $n$  tends to infinity), uniformly on  $f$  and  $\lambda$  as above.*

See Theorem 5.1 below.

For other work, following the current paper, see [H].

The rest of the paper is organized as follows. Definitions, notations and necessary preliminaries are given in Section 2. In Section 3 the expectation and variance of the major index are evaluated. Here Stanley's hook formula plays a crucial role. Results obtained in Section 3 are extended to general descent functions in Sections 4 and 5 by combining probabilistic arguments with combinatorial character formulas. We end the paper with remarks on connections of the statistics of descent functions with the spectra of so called 'good' elements in Hecke algebras.

## 2 Preliminaries

### 2.1 Young Tableaux

Let  $n$  be a positive integer. A *partition* of  $n$  is a vector of positive integers  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ , where  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$  and  $\lambda_1 + \dots + \lambda_k = n$ . For a partition  $\lambda = (\lambda_1, \dots, \lambda_k)$  define the *conjugate partition*  $\lambda' = (\lambda'_1, \dots, \lambda'_t)$  by letting  $\lambda'_i$  be the number of parts of  $\lambda$  that are  $\geq i$ .

The *dominance (partial) order* on partitions is defined as follows : For any two partitions of  $n$ ,  $\mu$  and  $\lambda$ ,  $\mu$  *dominates*  $\lambda$  if and only if for any  $i$   $\sum_{j=0}^i \lambda_j \leq \sum_{j=0}^i \mu_j$ .

For example,  $\lambda = (4, 4, 2, 1)$  is a partition of 11. Then  $\lambda' = (4, 3, 2, 2)$ , and  $\lambda$  dominates  $\lambda'$ .

The set  $\{(i, j) \mid i, j \in \mathbf{Z}, 0 < i \leq k, 0 < j \leq \lambda_i\}$  is called the *Young diagram* of shape  $\lambda$ .  $(i, j)$  is the *cell* in row  $i$  and column  $j$ . The diagram of the conjugate shape  $\lambda'$  may be obtained from the diagram of shape  $\lambda$  by interchanging rows and columns.

A *Young tableau* of shape  $\lambda$  is obtained by inserting the integers  $1, 2, \dots, n$  as *entries* in the cells of the Young diagram of shape  $\lambda$ , allowing no repetitions. A *standard Young tableau* of shape  $\lambda$  is a Young tableau whose entries increase along rows and columns.

We shall draw Young tableaux as in the following example.

**Example 1.**

$$\begin{array}{ccccc} 1 & 3 & 4 & 6 & 9 \\ 2 & 7 & 8 & 12 & \\ 5 & 11 & & & \\ 10 & & & & \end{array}$$

The *hook length* of a cell  $(i, j)$  in the diagram of shape  $\lambda$  is defined by

$$h_{i,j} := \lambda_i + \lambda'_j - i - j + 1.$$

Denote by  $f^\lambda$  the number of standard Young tableaux of shape  $\lambda$ . A famous combinatorial formula describes this number in term of hook lengths.

**The Frame-Robinson-Thrall Hook Formula.** [Sa, Theorem 3.1.2]

$$f^\lambda = \frac{n!}{\prod_{(i,j) \in \lambda} h_{i,j}}.$$

## 2.2 Descents

A *descent* in a standard Young tableau  $T$  is an entry  $i$  such that  $i + 1$  is strictly south (and weakly west) of  $i$ . Denote the set of all descents in  $T$  by  $D(T)$ . The *descent number* and the *major index* (for tableaux) are defined as follows :

$$des(T) := \sum_{i \in D(T)} 1 ; \quad maj(T) := \sum_{i \in D(T)} i.$$

**Example 2.** Let  $T$  be the standard Young tableau drawn in Example 1. Then  $D(T) = \{1, 4, 6, 9\}$ ,  $des(T) = 4$ , and  $maj(T) = 1 + 4 + 6 + 9 = 20$ .

The following theorem describes the generating function for the major index of standard Young tableaux.

**The Stanley Hook Formula** [St, Corollary 21.5]

$$\sum_{\text{shape}(T)=\lambda} q^{\text{maj}(T)} = q^{\sum_i (i-1)\lambda_i} \cdot \frac{\prod_{k=1}^n [k]_q}{\prod_{(i,j) \in \lambda} [h_{ij}]_q},$$

where the sum is taken over all standard Young tableaux of shape  $\lambda$ ,  $h_{ij}$  are the hook lengths in the diagram of  $\lambda$ , and for any positive integer  $m$

$$[m]_q := 1 + q + q^2 + \dots + q^{m-1}.$$

For  $q = 1$  this formula reduces to the Frame-Robinson-Thrall hook formula for the number of standard Young tableaux of a given shape. No such formula is known for the descent number of tableaux.

### 2.3 Characters

A (complex) representation of a group  $G$  is a homomorphism  $\rho : G \rightarrow GL_n(\mathbf{C})$ . The character  $\chi^\rho : G \rightarrow \mathbf{C}$  is the trace of  $\rho(g)$ ,  $g \in G$ . By definition, the character is a class function on the group (i.e., invariant under conjugation). An *irreducible* representation is a representation which has no nontrivial subspace invariant under all  $\rho(g)$ ,  $g \in G$ .

The conjugacy classes of the symmetric group  $S_n$  are described by their cycle type; thus, by the partitions of  $n$ . The irreducible representations of  $S_n$  are also indexed by these partitions. See e.g. [Sa].

Let  $\lambda$  and  $\mu$  be partitions of  $n$ . Denote by  $\chi_\mu^\lambda$  the value at a conjugacy class of cycle type  $\mu$  of the character of the irreducible representation indexed by  $\lambda$ .

The following combinatorial formula represents the irreducible characters of  $S_n$  in terms of descents of standard Young tableaux. This formula is a special case of [Ro1, Theorem 4].

**Theorem 2.1.**

$$\chi_\mu^\lambda = \sum_{\text{shape}(T)=\lambda} \text{weight}_\mu(T),$$

where the sum is taken over all standard tableaux of shape  $\lambda$ , and the weight  $\text{weight}_\mu(T) \in \{\pm 1, 0\}$  is defined as follows :

$$\text{weight}_\mu(T) := \prod_{\substack{1 \leq i \leq k \\ i \notin B(\mu)}} f_\mu(i, T),$$

where  $B(\mu) = \{\mu_1, \mu_1 + \mu_2, \dots, \mu_1 + \dots + \mu_t\}$ , and

$$f_\mu(i, T) := \begin{cases} -1 & i \in D(T); \\ 0 & i \notin D(T), i+1 \in D(T) \text{ and } i+1 \notin B(\mu); \\ 1 & \text{otherwise.} \end{cases}$$

### 3 Major Index

In this section we apply Stanley's hook formula (see Subsection 1.2) to evaluate the expectation and variance of the major index of random Young tableaux of a given shape.

**Proposition 3.1** *Let  $\lambda$  be a fixed partition of  $n$ , and let  $E_\lambda[\text{maj}]$  be the expected value of the major index on random standard Young tableaux of shape  $\lambda$ . Then*

$$E_\lambda[\text{maj}] = \frac{1}{2} \left[ \binom{n}{2} - \sum_i \binom{\lambda_i}{2} + \sum_j \binom{\lambda'_j}{2} \right].$$

**Proof.** Let

$$f(q) := \sum_{\text{shape}(T)=\lambda} q^{\text{maj}(T)}.$$

Then

$$q \cdot f'(q) = \sum_{\text{shape}(T)=\lambda} \text{maj}(T) \cdot q^{\text{maj}(T)}.$$

Hence

$$(3.1) \quad (\log f(q))'|_{q=1} = \frac{f'(q)}{f(q)} \Big|_{q=1} = \frac{\sum_{\text{shape}(T)=\lambda} \text{maj}(T)}{\sum_{\text{shape}(T)=\lambda} 1} = E_\lambda[\text{maj}].$$

In order to evaluate the expected value of the major index we need the following elementary limit :

$$(3.2) \quad \lim_{q \rightarrow 1} \frac{[m]_q'}{[m]_q} = \frac{\binom{m}{2}}{m} = \frac{m-1}{2}.$$

Substituting Stanley's hook formula in (3.1), and using (3.2) we obtain

$$(3.3) \quad E_\lambda[maj] = (\log f(q))'|_{q=1} = \sum_i (i-1)\lambda_i + \sum_{k=1}^n \frac{k-1}{2} - \sum_{(i,j) \in \lambda} \frac{h_{ij}-1}{2}.$$

Note that

$$(3.4) \quad \sum_i (i-1)\lambda_i = \sum_j \binom{\lambda'_j}{2}$$

and

$$(3.5) \quad \sum_{(i,j) \in \lambda} \frac{h_{ij}-1}{2} = \frac{1}{2} \left[ \sum_i \binom{\lambda_i}{2} + \sum_j \binom{\lambda'_j}{2} \right].$$

Substituting (3.4) and (3.5) in (3.3) completes the proof.  $\square$

**Proposition 3.2** *Let  $\lambda$  be a fixed partition of  $n$ , and let  $\text{Var}_\lambda[maj]$  be the variance of the major index on random standard Young tableaux of shape  $\lambda$ . Then*

$$\text{Var}_\lambda[maj] = \frac{1}{12} \left[ \sum_{k=1}^n k^2 - \sum_{(i,j) \in \lambda} h_{ij}^2 \right].$$

**Proof.** Let  $f$  denote, as in the previous proof, the generating function of the major index of Young tableaux of a given shape. Then

$$\sum_{\text{shape}(T)=\lambda} (\text{maj}(T))^2 \cdot q^{\text{maj}(T)} = q^2 \cdot f''(q) + q \cdot f'(q).$$

Hence

$$(3.6) \quad \begin{aligned} \text{Var}_\lambda[maj] &= E_\lambda[maj^2] - (E_\lambda[maj])^2 = \\ &= \left[ \frac{q^2 f'' + q f'}{f} - \left( q \frac{f'}{f} \right)^2 \right] \Big|_{q=1} = \left[ q^2 \left( \frac{f'}{f} \right)' + q \frac{f'}{f} \right] \Big|_{q=1}. \end{aligned}$$

Now

$$\frac{f'}{f} = (\log f)' = \sum_i (i-1)\lambda_i \cdot q^{-1} + \sum_{k=1}^n \frac{[k]_q'}{[k]_q} - \sum_{(i,j) \in \lambda} \frac{[h_{ij}]_q'}{[h_{ij}]_q}$$

and

$$\begin{aligned}
\lim_{q \rightarrow 1} \left( \frac{[m]_q'}{[m]_q} \right)' &= \lim_{q \rightarrow 1} \left[ \frac{[m]_q''}{[m]_q} - \left( \frac{[m]_q'}{[m]_q} \right)^2 \right] = \\
&= \lim_{q \rightarrow 1} \left[ \frac{\sum_{k=2}^m (k-1)(k-2)q^{k-3}}{\sum_{k=1}^m q^{k-1}} - \left( \frac{\sum_{k=1}^m (k-1)q^{k-2}}{\sum_{k=1}^m q^{k-1}} \right)^2 \right] = \\
&= \frac{\sum_{k=2}^m (k-1)(k-2)}{m} - \left( \frac{\sum_{k=1}^m (k-1)}{m} \right)^2 = \frac{2\binom{m}{3}}{m} - \left( \frac{m-1}{2} \right)^2 = \frac{(m-1)(m-5)}{12}.
\end{aligned}$$

Hence

$$(3.7) \quad \lim_{q \rightarrow 1} \left( \frac{f'}{f} \right)' = - \sum_i (i-1)\lambda_i + \sum_{k=1}^n \frac{(k-1)(k-5)}{12} - \sum_{(i,j) \in \lambda} \frac{(h_{ij}-1)(h_{ij}-5)}{12}.$$

Substituting (3.7) and (3.3) into the right hand side of (3.6) we obtain the desired result.  $\square$

**Corollary 3.3** *Let  $0 < \delta < 1$  and  $\varepsilon > 0$  be fixed constants, and let  $\lambda$  be a partition of  $n$  with  $\lambda_1 \leq \delta n$ . Then*

$$maj(T) = (1 + O(n^{-\frac{1}{2}+\varepsilon}))E_\lambda[maj]$$

*almost surely (i.e., with probability tending to 1 as  $n$  tends to infinity) uniformly on  $\lambda$  as above.*

**Proof.** By Chebyshev's inequality [Fe, (6.2)]

$$\Pr(|maj(T) - E_\lambda[maj]| \geq tE_\lambda[maj]) \leq \frac{1}{t^2} \cdot \frac{\text{Var}_\lambda[maj]}{E_\lambda[maj]^2}.$$

Now, by Proposition 3.2

$$0 \leq \text{Var}_\lambda[maj] \leq \frac{1}{12} \sum_{k=1}^n k^2 = O(n^3).$$

In order to bound  $E_\lambda[maj]$  from below, note that

$$(3.8) \quad \sum_i \binom{\lambda_i}{2} - \sum_j \binom{\lambda_j'}{2} = \sum_{(i,j) \in \lambda} (j-i).$$



Thus this expression is a monotone increasing function of  $\lambda$  with respect to the dominance order of partitions (see also [Su]). So, under the restriction  $\lambda_1 \leq \delta n$  (we may assume that  $\delta > \frac{1}{2}$ ) this expression is maximized when  $\lambda_1 = \delta n, \lambda_2 = (1 - \delta)n$ . Hence, by Proposition 3.1 :

$$\begin{aligned} E_\lambda[maj] &= \frac{1}{2} \left[ \binom{n}{2} - \sum_i \binom{\lambda_i}{2} + \sum_j \binom{\lambda'_j}{2} \right] = \frac{1}{2} \left[ \binom{n}{2} - \sum_{(i,j) \in \lambda} (j - i) \right] \geq \\ &\geq \frac{1}{2} \left[ \binom{n}{2} - \binom{\delta n}{2} - \binom{(1 - \delta)n}{2} \right] = \Omega(n^2). \end{aligned}$$

Therefore,

$$\frac{1}{t^2} \cdot \frac{\text{Var}_\lambda[maj]}{E_\lambda[maj]^2} \leq O\left(\frac{1}{t^2 n}\right) = O(n^{-2\varepsilon}),$$

provided that  $\frac{1}{t} = O(n^{1/2-\varepsilon})$ . We conclude that for such  $\lambda$  and  $t$

$$(1 - t)E_\lambda[maj] < maj(T) < (1 + t)E_\lambda[maj]$$

with probability tending to 1 as  $n$  tends to infinity. □

## 4 Expectation of Descent Functions

In this section we generalize Proposition 3.1 to an arbitrary descent function. This is done by combining probabilistic arguments together with combinatorial character formulas.

Recall the definition of descent functions from Section 1.

**Theorem 4.1** *Let  $\lambda$  be a fixed partition of  $n$ , let  $f : \mathbf{N} \rightarrow \mathbf{R}$  be an arbitrary function, and let  $E_\lambda[d_f]$  be the expected value of  $d_f$  on random standard Young tableaux of shape  $\lambda$ . Then*

$$E_\lambda[d_f] = c(\lambda) \cdot \sum_{i=1}^{n-1} f(i).$$

where  $c(\lambda) := [\binom{n}{2} - \sum_i \binom{\lambda_i}{2} + \sum_j \binom{\lambda'_j}{2}] / n(n - 1)$ .

Substituting  $f(i) = 1$  and  $f(i) = i$  in Theorem 4.1 gives the expectations for the descent number and for the major index of a random tableau, respectively. In particular, Proposition 3.1 is a special case of Theorem 4.1.

To prove Theorems 4.1 and 5.1 we shall use a variant of the character formula given in Theorem 2.1.

For a fixed  $1 \leq i < n$  define the  $i$ -weight

$$\text{weight}_{(21\dots 1)}^i(T) := \begin{cases} -1, & \text{if } i \in D(T) ; \\ 1, & \text{if } i \notin D(T) . \end{cases}$$

For a fixed  $1 \leq i < n - 1$  define another  $i$ -weight

$$\text{weight}_{(31\dots 1)}^i(T) := \begin{cases} -1, & \text{if } i \in D(T) \text{ and } i+1 \notin D(T) ; \\ 1, & \text{if } i, i+1 \in D(T) \text{ or } i, i+1 \notin D(T) ; \\ 0, & \text{if } i \notin D(T) \text{ and } i+1 \in D(T) . \end{cases}$$

For a fixed pair  $1 \leq i < j < n$  with  $j - i > 1$  define the  $ij$ -weight

$$\text{weight}_{(221\dots 1)}^{ij}(T) := \begin{cases} 1, & \text{if } i, j \in D(T) \text{ or } i, j \notin D(T) ; \\ -1, & \text{otherwise .} \end{cases}$$

The  $i$ - (or  $ij$ -) weight of a standard tableau  $T$  depends on  $i$  (or  $i, j$ ). However, the sum of  $i$ -weights (or  $ij$ -weights) over all standard tableaux of shape  $\lambda$  is independent of  $i$  and  $j$  and gives the corresponding character:

**Lemma 4.2** *For any partition  $\lambda$  of  $n$ ,*

$$(i) \quad \chi_{(21\dots 1)}^\lambda = \sum_{\text{shape}(T)=\lambda} \text{weight}_{(21\dots 1)}^i(T) \quad (1 \leq i < n)$$

$$(ii) \quad \chi_{(31\dots 1)}^\lambda = \sum_{\text{shape}(T)=\lambda} \text{weight}_{(31\dots 1)}^i(T) \quad (1 \leq i < n - 1)$$

$$(iii) \quad \chi_{(221\dots 1)}^\lambda = \sum_{\text{shape}(T)=\lambda} \text{weight}_{(221\dots 1)}^{ij}(T) \quad (1 \leq i < j - 1 < n - 1)$$

For proofs and more details see [Ro1, Section 7].

Recall also that

$$\chi_{(1\dots 1)}^\lambda = \sum_{\text{shape}(T)=\lambda} 1 = f^\lambda.$$

**Proof of Theorem 4.1.** Let  $T$  be a random standard Young tableau of shape  $\lambda$ . For  $1 \leq i < n$ , let  $X_i$  be the random variable defined by

$$X_i := \begin{cases} 1, & \text{if } i \in D(T) ; \\ 0, & \text{otherwise .} \end{cases}$$

Then

$$(4.1) \quad E_\lambda[d_f] = E_\lambda \left[ \sum_{i=1}^{n-1} f(i) X_i \right] = \sum_{i=1}^{n-1} f(i) E_\lambda[X_i].$$

Now, by definition

$$\text{weight}_{(21\dots 1)}^i = 1 - 2X_i$$

and therefore, by Lemma 4.2(i)

$$(4.2) \quad 1 - 2E_\lambda[X_i] = E_\lambda[\text{weight}_{(21\dots 1)}^i] = \chi_{(21\dots 1)}^\lambda / \chi_{(1\dots 1)}^\lambda.$$

In particular, note that  $E_\lambda[X_i]$  is independent of  $i$  (See [St, Prop. 7.19.9]).

A classical formula of Frobenius shows that

$$(4.3) \quad \chi_{(21\dots 1)}^\lambda / \chi_{(1\dots 1)}^\lambda = \frac{1}{\binom{n}{2}} \left[ \sum_i \binom{\lambda_i}{2} - \sum_j \binom{\lambda'_j}{2} \right].$$

See [I].

Combining (4.1), (4.2) and (4.3) completes the proof. □

## 5 Concentration

In this section Corollary 3.3 is generalized to descent functions, satisfying certain mild conditions.

Recall that a function  $f : \mathbf{N} \rightarrow \mathbf{R}$  has *strictly polynomial growth* if there exist constants  $0 < c_1 < c_2$  and  $\alpha > 0$ , such that  $c_1 \leq \frac{f(n)}{n^\alpha} \leq c_2$  for  $n$  large enough.

**Theorem 5.1** *Let  $0 < \delta < 1$  and  $\varepsilon > 0$  be fixed constants, and let  $\lambda$  be a partition of  $n$  with  $\lambda_1 \leq \delta n$ . Then for any function  $f$  with strictly polynomial growth, and for a random standard Young tableau  $T$  of shape  $\lambda$ ,*

$$d_f(T) = (1 + O(n^{-\frac{1}{2} + \varepsilon})) E_\lambda[d_f]$$

almost surely (i.e., with probability tending to 1 as  $n$  tends to infinity), uniformly on  $f$  and  $\lambda$  as above.

This holds, in particular, if  $d_f$  is either descent number or major index.

Theorem 5.1 is proved by estimating the asymptotic behavior of the variance. This is done by expressing the variance in terms of  $S_n$ -characters evaluated at “small” conjugacy classes, and showing that  $\text{Var}_\lambda[d_f]/E_\lambda[d_f]^2$  is independent of the function  $f$ , up to a multiplicative constant.

Denote

$$\begin{aligned} r_2^\lambda &:= \chi_{21\dots 1}^\lambda / \chi_{1\dots 1}^\lambda, \\ r_3^\lambda &:= \chi_{31\dots 1}^\lambda / \chi_{1\dots 1}^\lambda, \\ r_{22}^\lambda &:= \chi_{221\dots 1}^\lambda / \chi_{1\dots 1}^\lambda. \end{aligned}$$

These are the values of the normalized irreducible character corresponding to  $\lambda$  at conjugacy classes of types  $(21\dots 1)$ ,  $(31\dots 1)$  and  $(221\dots 1)$  respectively.

The following lemma plays a crucial role in the proof of Theorem 5.1.

**Lemma 5.2.**

$$(i) \quad \Pr[X_i = 1] = \frac{1}{2}(1 - r_2^\lambda) \quad (1 \leq i < n).$$

$$(ii) \quad \Pr[X_i = 1 \text{ and } X_{i+1} = 1] = \frac{1}{2}(1 - r_2^\lambda) - \frac{1}{3}(1 - r_3^\lambda) \quad (1 \leq i < n - 1).$$

$$(iii) \quad \Pr[X_i = 1 \text{ and } X_j = 1] = \frac{1}{2}(1 - r_2^\lambda) - \frac{1}{4}(1 - r_{22}^\lambda) \quad (1 \leq i < j - 1 < n - 1),$$

where the probability  $\Pr[\cdot]$  is taken in the probability space of all standard Young tableaux of a given shape, defined in Section 1.

**Proof.**

(i) Since  $X_i$  is a 0-1 variable,

$$E_\lambda[X_i] = \Pr[X_i = 1].$$

Using equation (4.2) and the definition of  $r_2^\lambda$ , the desired result follows.

(ii) By Lemma 4.2(ii), for  $1 \leq i < n - 1$  :

$$r_3^\lambda = \Pr[X_i = 1 \wedge X_{i+1} = 1] + \Pr[X_i = 0 \wedge X_{i+1} = 0] - \Pr[X_i = 1 \wedge X_{i+1} = 0].$$

Therefore,

(5.1)

$$r_3^\lambda = (1 - \Pr[X_i = 1 \wedge X_{i+1} = 0] - \Pr[X_i = 0 \wedge X_{i+1} = 1]) - \Pr[X_i = 1 \wedge X_{i+1} = 0].$$

Now

$$\Pr[X_i = 1 \wedge X_{i+1} = 0] = \Pr[X_i = 1] - \Pr[X_i = 1 \wedge X_{i+1} = 1],$$

$$\Pr[X_i = 0 \wedge X_{i+1} = 1] = \Pr[X_{i+1} = 1] - \Pr[X_i = 1 \wedge X_{i+1} = 1],$$

and therefore, using (i) above

$$\begin{aligned} \Pr[X_i = 1 \wedge X_{i+1} = 0] &= \Pr[X_i = 0 \wedge X_{i+1} = 1] = \\ &= \frac{1}{2}(1 - r_2^\lambda) - \Pr[X_i = 1 \wedge X_{i+1} = 1]. \end{aligned}$$

Thus, from (5.1):

$$\begin{aligned} 1 - r_3^\lambda &= 2\Pr[X_i = 1 \wedge X_{i+1} = 0] + \Pr[X_i = 0 \wedge X_{i+1} = 1] = \\ &= \frac{3}{2}(1 - r_2^\lambda) - 3\Pr[X_i = 1 \wedge X_{i+1} = 1], \end{aligned}$$

and (ii) follows.

(iii) By Lemma 4.2(iii), for any  $1 \leq i < j - 1 < n - 1$

$$\begin{aligned} r_{22}^\lambda &= \Pr[X_i = 1 \wedge X_j = 1] + \Pr[X_i = 0 \wedge X_j = 0] - \\ &\quad - \Pr[X_i = 0 \wedge X_j = 1] - \Pr[X_i = 1 \wedge X_j = 0]. \end{aligned}$$

Continue as in the proof of (ii) above.

□

Denote

$$P_{2,\lambda} := \Pr[X_i = 1],$$

$$P_{3,\lambda} := \Pr[X_i = 1 \text{ and } X_{i+1} = 1],$$

$$P_{22,\lambda} := \Pr[X_i = 1 \text{ and } X_j = 1] \quad (j - i > 1).$$

By Lemma 5.2, these probabilities are well defined (independent of  $i$  and  $j$ ).

**Proof of Theorem 5.1.** By Chebyshev's inequality

$$\Pr(|d_f - E_\lambda[d_f]| \geq tE_\lambda[d_f]) \leq \frac{1}{t^2} \cdot \frac{\text{Var}_\lambda[d_f]}{E_\lambda[d_f]^2}.$$

In order to prove Theorem 5.1 it suffices to give an effective upper bound on  $\text{Var}_\lambda[d_f]/E_\lambda[d_f]^2$ .

For a random tableau  $T$  of shape  $\lambda$ ,

$$d_f(T) = \sum_{i=1}^{n-1} f(i)X_i.$$

Thus,

(5.2)

$$\begin{aligned} \text{Var}_\lambda[d_f] &= E_\lambda[d_f^2] - E_\lambda[d_f]^2 = E_\lambda \left[ \left( \sum_{i=1}^{n-1} X_i f(i) \right)^2 \right] - \left( E_\lambda \left[ \sum_{i=1}^{n-1} X_i f(i) \right] \right)^2 = \\ &= \sum_{i=1}^{n-1} E_\lambda[X_i^2] f(i)^2 + 2 \sum_{i=1}^{n-2} E_\lambda[X_i X_{i+1}] f(i) f(i+1) + \\ &\quad + 2 \sum_{j-i>1} E_\lambda[X_i X_j] f(i) f(j) - \left( \sum_{i=1}^{n-1} E_\lambda[X_i] f(i) \right)^2. \end{aligned}$$

Since  $X_i$  is a 0 – 1 variable

$$(5.3) \quad E_\lambda[X_i^2] = E_\lambda[X_i] = P_{2,\lambda} \quad (1 \leq i < n),$$

$$(5.4) \quad E_\lambda[X_i X_{i+1}] = P_{3,\lambda} \quad (1 \leq i < n-1),$$

and

$$(5.5) \quad E_\lambda[X_i X_j] = P_{22,\lambda} \quad (j-i > 1).$$

Substituting (5.3)-(5.5) into the right hand side of (5.2) we obtain

$$\begin{aligned} \text{Var}_\lambda[d_f] &= P_{2,\lambda} \sum_{i=1}^{n-1} f(i)^2 + 2P_{3,\lambda} \sum_{i=1}^{n-2} f(i) f(i+1) + \\ &\quad + 2P_{22,\lambda} \sum_{j-i>1} f(i) f(j) - P_{2,\lambda}^2 \left( \sum_{i=1}^{n-1} f(i) \right)^2. \end{aligned}$$

Denote

$$\begin{aligned}\Sigma_1 &:= \sum_{i=1}^{n-1} f(i), \\ \Sigma_2 &:= \sum_{i=1}^{n-1} f(i)^2, \\ \Sigma_3 &:= 2 \sum_{i=1}^{n-2} f(i)f(i+1).\end{aligned}$$

Obviously

$$2 \sum_{j-i>1} f(i)f(j) = \Sigma_1^2 - \Sigma_2 - \Sigma_3,$$

and therefore

$$\begin{aligned}\text{Var}_\lambda[d_f] &= P_{2,\lambda}\Sigma_2 + P_{3,\lambda}\Sigma_3 + P_{22,\lambda}(\Sigma_1^2 - \Sigma_2 - \Sigma_3) - P_{2,\lambda}^2\Sigma_1^2 = \\ &= (P_{2,\lambda} - P_{22,\lambda})\Sigma_2 + (P_{3,\lambda} - P_{22,\lambda})\Sigma_3 + (P_{22,\lambda} - P_{2,\lambda}^2)\Sigma_1^2.\end{aligned}$$

Also

$$E_\lambda[d_f] = P_{2,\lambda}\Sigma_1,$$

and consequently, using Lemma 5.2(i)-(iii),

$$(5.6) \quad \frac{\text{Var}_\lambda[d_f]}{E_\lambda[d_f]^2} = \frac{1 - r_{22}^\lambda}{(1 - r_2^\lambda)^2} \cdot \frac{\Sigma_2}{\Sigma_1^2} + \frac{4r_3^\lambda - 3r_{22}^\lambda - 1}{3(1 - r_2^\lambda)^2} \cdot \frac{\Sigma_3}{\Sigma_1^2} + \frac{r_{22}^\lambda - (r_2^\lambda)^2}{(1 - r_2^\lambda)^2}.$$

$f$  has a strictly polynomial growth. It follows that

$$(5.7) \quad \frac{\Sigma_2}{\Sigma_1^2} = O(n^{-1}) \quad \text{and} \quad \frac{\Sigma_3}{\Sigma_1^2} = O(n^{-1})$$

where the constants in  $O(\cdot)$  depend only on  $c_1, c_2$  and  $\alpha$ .

It follows from formula (4.3) and the proof of Corollary 3.3 that for any fixed  $\frac{1}{2} < \delta < 1$

$$\begin{aligned}\max_{\lambda_1 \leq \delta n} r_2^\lambda &= \max_{\lambda_1 \leq \delta n} \frac{1}{\binom{n}{2}} \left[ \sum_i \binom{\lambda_i}{2} - \sum_j \binom{\lambda'_j}{2} \right] \leq \\ &\leq \frac{1}{\binom{n}{2}} \cdot \left[ \binom{\delta n}{2} + \binom{(1-\delta)n}{2} \right] \leq \delta^2 + (1-\delta)^2,\end{aligned}$$

where the maximum is taken over all partitions  $\lambda$  of  $n$  with  $\lambda_1 \leq \delta n$ .

Hence

$$(5.8) \quad \frac{1}{(1 - r_2^\lambda)^2} \leq \frac{1}{2\delta(1 - \delta)}.$$

The absolute value of a normalized character of a finite group is bounded above by 1. Combining this elementary fact with (5.8) implies that there exist constants  $c_1(\delta), c_2(\delta)$  independent of  $n$ , so that

$$(5.9) \quad c_1(\delta) \leq \frac{1 - r_{22}^\lambda}{(1 - r_2^\lambda)^2} \leq c_2(\delta) \quad \text{and} \quad c_1(\delta) \leq \frac{4r_3^\lambda - 3r_{22}^\lambda - 1}{3(1 - r_2^\lambda)^2} \leq c_2(\delta).$$

To complete the proof it still remains to estimate the asymptotics of  $r_{22}^\lambda - (r_2^\lambda)^2$ .

Using the classical Frobenius character formula, it may be shown that

$$r_{22}^\lambda = \frac{4}{(n-2)(n-3)} + \frac{1}{\binom{n}{2,2,n-4}} \cdot \left[ \left( \sum_{(i,j) \in \lambda} (j-i) \right)^2 - 3 \sum_{(i,j) \in \lambda} (j-i)^2 \right].$$

See [I] and [Su].

Using this formula, (4.3) and (3.8) one obtains

$$(5.10) \quad |r_{22}^\lambda - (r_2^\lambda)^2| \leq \frac{4}{(n-2)(n-3)} + \left[ \frac{\binom{n}{2}^2}{\binom{n}{2,2,n-4}} - 1 \right] (r_2^\lambda)^2 + \\ + \frac{3}{\binom{n}{2,2,n-4}} \sum_{(i,j) \in \lambda} (j-i)^2 \leq O(n^{-2}) + O(n^{-1})(r_2^\lambda)^2 + O(n^{-1}) = O(n^{-1}).$$

Substituting (5.7)-(5.10) into the right hand side of (5.6) shows that

$$\frac{\text{Var}_\lambda[d_f]}{E_\lambda[d_f]^2} = O(n^{-1}).$$

This completes the proof. □



## 6 Exponents in Hecke Algebras

Surprisingly, the expectations which appear in Proposition 3.1 and Theorem 4.1 turn out to be the exponents of  $q$  in eigenvalues of irreducible Hecke algebra representations. In particular,  $E_\lambda[\text{des}]$  and  $E_\lambda[\text{maj}]$  are the exponents of  $q$  for the Hecke algebra elements corresponding to a Coxeter element and the longest element of  $S_n$ , respectively. This follows from a well-known result of Benson and Curtis [BC].

The Hecke algebra  $\mathcal{H}_n(q)$  of type  $A$  is the algebra over  $\mathbf{F} := \mathbf{C}(q^{\frac{1}{2}})$  generated by  $n - 1$  generators  $T_1, \dots, T_{n-1}$ , satisfying the Moore-Coxeter relations

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad (1 \leq i < n - 1)$$

and

$$T_i T_j = T_j T_i \quad \text{if } |i - j| > 1.$$

as well as the following “deformed involution” relation:

$$T_i^2 = (1 - q)T_i + q \quad (1 \leq i < n).$$

Note that the third relation is slightly non-standard. This is done in order to have a more elegant formulation of Proposition 6.1.

Let  $w_0 := (1, n)(2, n-1) \cdots$  be the longest permutation in the symmetric group  $S_n$ , and  $c_n := (12 \dots n)$  be a full cycle (also known as a Coxeter element).

### Proposition 6.1

- (i) *The eigenvalues of  $T_{w_0}$  in the irreducible representation indexed by  $\lambda$  are  $\pm q^{E_\lambda[\text{maj}]}$ .*
- (ii) *The eigenvalues of  $T_{c_n}$  in the irreducible representation indexed by  $\lambda$  are*

$$\omega^{\text{maj}(T)} q^{E_\lambda[\text{des}]}$$

where  $T$  ranges over all standard tableaux of shape  $\lambda$ , and  $\omega = e^{\frac{2\pi i}{n}}$ .

**Proof.** For self containment of the paper we recall the proof of [BC, corrections and additions]. See also [GM, Lemma 4.1]. It is well known that  $T_{w_0}^2$  lies in the center of  $\mathcal{H}_n(q)$ . Hence, denoting by  $\rho^\lambda$  the irreducible representation of  $\mathcal{H}_n(q)$  indexed by  $\lambda$ ,  $\rho^\lambda(T_{w_0}^2)$  is a scalar operator.

On the other hand for each generator  $T_i$ ,  $1 \leq i < n$ ,  $\rho^\lambda(T_i)$  has two eigenvalues: 1 and  $-q$ , with multiplicities  $\frac{1}{2}(f^\lambda + \chi_2^\lambda)$  and  $\frac{1}{2}(f^\lambda - \chi_2^\lambda)$  respectively. Here  $f^\lambda$  is the degree of  $\rho^\lambda$  and  $\chi_2^\lambda = \text{tr} \rho^\lambda(T_i)|_{q=1}$ . Hence

$$\det \rho^\lambda(T_i) = (-q)^{\frac{1}{2}(f^\lambda - \chi_2^\lambda)} = \pm q^{\frac{1}{2}(f^\lambda - \chi_2^\lambda)}.$$

$T_{w_0}$  is a product of  $\binom{n}{2}$  generators  $T_i$ . Hence

$$\det \rho^\lambda(T_{w_0}^2) = \det \rho^\lambda(T_i)^{n(n-1)} = q^{\binom{n}{2}(f^\lambda - \chi_2^\lambda)}.$$

This shows that the eigenvalues of the scalar operator  $\rho^\lambda(T_{w_0}^2)$  are all equal to  $q^{\binom{n}{2} \frac{f^\lambda - \chi_2^\lambda}{f^\lambda}}$ , and those of  $T_{w_0}$  are

$$\pm q^{\binom{n}{2} \frac{f^\lambda - \chi_2^\lambda}{2f^\lambda}} = \pm q^{\binom{n}{2} \frac{1 - r_2^\lambda}{2}} = \pm q^{E_\lambda[maj]}.$$

The last equality follows from Proposition 3.1 and formula (4.3).

To prove the second part of Proposition 6.1 recall that

$$T_{c_n}^n = T_{w_0}^2.$$

So, the eigenvalues of  $T_{c_n}$  are the complex  $n$ -th roots of the eigenvalues of  $\rho^\lambda(T_{w_0}^2)$ . Combining this fact with Theorem 4.1 gives the corresponding exponent of  $q$ . For a calculation of the exponents of  $\omega$  see [Ste].  $\square$

Proposition 6.1 may be useful in the study of probabilistic interpretations of the Hecke algebra, and of related asymmetric random walks.

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